

A NEW WEBER TYPE INTEGRAL EQUATION RELATED TO THE WEBER-TITCHMARSH PROBLEM

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ABSTRACT. We derive solvability conditions and closed-form solution for the Weber type integral equation, related to the familiar Weber-Orr integral transforms and the old Weber-Titchmarsh problem (posed in *Proc. Lond. Math. Soc.* **22**(2) (1924), pp.15, 16), recently solved by the author. Our method involves properties of the inverse Mellin transform of integrable functions. The Mellin-Parseval equality and some integrals, involving the Gauss hypergeometric function are used.

Recently, the author gave solvability conditions and closed-form solution for the classical Weber equation [7]

$$\int_0^\infty C_\nu(x\xi, a\xi)g(\xi)d\xi = f(x), \quad (1)$$

where $f(x)$ is a given function on $[a, \infty)$, $a > 0$, $g(x), x \in \mathbb{R}_+$ should be determined and the kernel

$$C_\nu(\alpha, \beta) = J_\nu(\alpha)Y_\nu(\beta) - Y_\nu(\alpha)J_\nu(\beta) \quad (2)$$

involves Bessel functions of the first and second kind $J_\nu(z), Y_\nu(z)$, $\nu \in \mathbb{C}$ [1], Vol. II. It was solved formally by Titchmarsh in 1924 and posed as an open problem (see [3], p. 15.) to describe a class of complex-valued functions $g(x)$, $x \in \mathbb{R}_+$, which can be expanded in terms of the following repeated integral

$$g(x) = \frac{x}{J_\nu^2(ax) + Y_\nu^2(ax)} \int_a^\infty C_\nu(xt, xa)t \int_0^\infty C_\nu(t\xi, a\xi)g(\xi)d\xi dt, \quad x > 0. \quad (3)$$

Expansion (3) is related to the familiar Weber-Orr integral expansions of an arbitrary function $f(x)$ as repeated integrals

$$f(x) = \int_0^\infty \frac{t C_\nu(xt, at)}{J_\nu^2(at) + Y_\nu^2(at)} \int_a^\infty C_\nu(\xi t, at)\xi f(\xi)d\xi dt, \quad (4)$$

$$f(x) = \int_a^\infty C_\nu(xt, xa)t \int_0^\infty \frac{C_\nu(t\xi, a\xi)}{J_\nu^2(a\xi) + Y_\nu^2(a\xi)} \xi f(\xi)d\xi dt, \quad (5)$$

which are different from (3). Our method is based on the use of the Mellin transform [4]. Precisely, the Mellin transform is defined in $L_{\mu,p}(\mathbb{R}_+)$, $1 < p \leq 2$ by the integral

$$f^*(s) = \int_0^\infty f(x)x^{s-1}dx, \quad (6)$$

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being convergent in mean with respect to the norm in $L_q(\mu - i\infty, \mu + i\infty)$, $q = p/(p-1)$. Moreover, the Parseval equality holds for $f \in L_{\mu,p}(\mathbb{R}_+)$, $g \in L_{1-\mu,q}(\mathbb{R}_+)$

$$\int_0^\infty f(x)g(x)dx = \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} f^*(s)g^*(1-s)ds. \quad (7)$$

The inverse Mellin transform is given accordingly

$$f(x) = \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} f^*(s)x^{-s}ds, \quad (8)$$

where the integral converges in mean with respect to the norm in $L_{\mu,p}(\mathbb{R}_+)$

$$\|f\|_{\mu,p} = \left(\int_0^\infty |f(x)|^p x^{\mu p-1} dx \right)^{1/p}. \quad (9)$$

In particular, letting $\mu = 1/p$ we get the usual space $L_1(\mathbb{R}_+)$. We will modify the definition of a special class of functions related to the Mellin transform (6) and its inversion (8), which was introduced in [5], [6]. Indeed, we have

Definition 1. Denote by $\mathcal{M}^{-1}(L_c)$ the space of functions $f(x)$, $x \in \mathbb{R}_+$, being representable by inverse Mellin transform (8) of integrable functions $F(s) \in L_1(c)$ on the vertical line $c = \{s \in \mathbb{C} : \mu = \text{Res} = c_0\}$.

The space $\mathcal{M}^{-1}(L_c)$ with the usual operations of addition and multiplication by scalar is a linear vector space. If the norm in $\mathcal{M}^{-1}(L_c)$ is introduced by the formula

$$\|f\|_{\mathcal{M}^{-1}(L_c)} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |F(c_0 + it)| dt, \quad (10)$$

then it becomes a Banach space.

Definition 2 . Let $\mu \neq 0$, $c_1, c_2 \in \mathbb{R}$ be such that $2\text{sign } c_1 + \text{sign } c_2 \geq 0$. By $\mathcal{M}_{c_1, c_2}^{-1}(L_c)$ we denote the space of functions $f(x)$, $x \in \mathbb{R}_+$, representable in the form (8), where $s^{c_2} e^{\pi c_1 |s|} F(s) \in L_1(c)$.

It is a Banach space with the norm

$$\|f\|_{\mathcal{M}_{c_1, c_2}^{-1}(L_c)} = \frac{1}{2\pi} \int_c e^{\pi c_1 |s|} |s^{c_2} F(s)| ds.$$

In particular, letting $c_1 = c_2 = 0$ we get the space $\mathcal{M}^{-1}(L_c)$. Moreover, it is easily seen the inclusion

$$\mathcal{M}_{d_1, d_2}^{-1}(L_c) \subseteq \mathcal{M}_{c_1, c_2}^{-1}(L_c)$$

when $2\text{sign}(d_1 - c_1) + \text{sign}(d_2 - c_2) \geq 0$.

Using this technique we proved the following

Theorem 1 [7]. Let $a > 0$, $v \in \mathbb{C}$, $0 < \text{Re } v < 1/2$, $g(x) \in \mathcal{M}_{0,1}^{-1}(L_c)$ with $c = \{s \in \mathbb{C} : -1 < \text{Re } s < 0\}$. Then for almost all $x > 0$ expansion (3) holds, where the inner and outer integrals are understood in the improper sense.

These results will be applied to solve the so-called Weber type integral equation

$$\int_0^\infty \varphi(\lambda) [J_v(x\lambda)Y_{v+1}(a\lambda) - Y_v(x\lambda)J_{v+1}(a\lambda)] d\lambda = f(x), \quad x > a > 0 \quad (11)$$

in the class $\mathcal{M}_{0,1}^{-1}(L_c)$. The key ingredient will be also properties for derivative of Bessel functions, namely, (see [1], Vol. II)

$$\left[x^{\mp v} \frac{d}{dx} x^{\pm v} \right] J_v(x) = \pm J_{v \mp 1}(x), \quad (12)$$

$$\left[x^{\mp \nu} \frac{d}{dx} x^{\pm \nu} \right] Y_{\nu}(x) = \pm Y_{\nu \mp 1}(x), \quad (13)$$

and the following integral, which is a direct consequence of relation (2.13.15.4) in [2], Vol. 2, namely,

$$\begin{aligned} F_{\nu}(x, s) &= \int_0^{\infty} \lambda^{-s} [J_{\nu}(x\lambda)Y_{\nu+1}(a\lambda) - Y_{\nu}(x\lambda)J_{\nu+1}(a\lambda)] d\lambda = \frac{2^{-s}a^{\nu+1}}{\pi x^{2-s+\nu}} \frac{\cos(\pi\nu)}{\Gamma(s/2)} \Gamma(-\nu-1) \Gamma(1+\nu-s/2) \\ &\quad \times {}_2F_1\left(1-\frac{s}{2}, 1+\nu-\frac{s}{2}; 2+\nu; \frac{a^2}{x^2}\right) - \frac{2^{-s}x^{\nu+s}}{\pi a^{1+\nu}} \frac{\Gamma(\nu+1)\Gamma(-s/2)}{\Gamma(1+\nu+s/2)} \\ &\quad \times {}_2F_1\left(-\nu-\frac{s}{2}, -\frac{s}{2}; -\nu; \frac{a^2}{x^2}\right) - \frac{2^{-s}a^{\nu+1}}{\pi x^{1+\nu-s}} \cos\left(\frac{\pi s}{2}\right) \frac{\Gamma(\nu+1-s/2)\Gamma(1-s/2)}{\Gamma(2+\nu)} \\ &\quad \times {}_2F_1\left(1+\nu-\frac{s}{2}, 1-\frac{s}{2}; 2+\nu; \frac{a^2}{x^2}\right), \quad -1 < \text{Re } s < 0, \quad x > a, \end{aligned} \quad (14)$$

where $\Gamma(z)$ is Euler's gamma-function and ${}_2F_1(a, b; c; x)$ is Gauss's hypergeometric function [1], Vol. 1, having an integral representation as the Euler integral

$${}_2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-1} (1-u)^{c-b-1} (1-xu)^{-a} du, \quad (15)$$

for instance, under conditions $\text{Re } a > 0$, $\text{Re } c > \text{Re } b > 0$, $x \in [0, 1)$. Moreover, representation (15) gives us the following uniform estimate for the Gauss function, which will be used below

$$\begin{aligned} |{}_2F_1(a, b; c; x)| &\leq \left| \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \right| \int_0^1 u^{\text{Re } b-1} (1-u)^{\text{Re}(c-b)-1} (1-xu)^{-\text{Re } a} du \\ &\leq (1-x)^{-\text{Re } a} B(\text{Re } b, \text{Re}(c-b)) \left| \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \right|, \end{aligned} \quad (16)$$

where $B(a, b)$ is Euler's beta-function [1], Vol. 1. Let $-1 < \text{Re } \nu < -1/2$. Then, denoting by the same letter C various positive constants, which can occur, we obtain

$$\begin{aligned} \left| {}_2F_1\left(1-\frac{s}{2}, 1+\nu-\frac{s}{2}; 2+\nu; \frac{a^2}{x^2}\right) \right| &\leq C x^{2-\text{Re } s} (x^2 - a^2)^{\text{Re } s/2-1} \left| \frac{\Gamma(2+\nu)}{\Gamma((2(1+\nu)-s)/2)\Gamma((2+s)/2)} \right|, \\ \left| {}_2F_1\left(-\nu-\frac{s}{2}, -\frac{s}{2}; -\nu; \frac{a^2}{x^2}\right) \right| &\leq C x^{-2\nu-\text{Re } s} (x^2 - a^2)^{\nu+\text{Re } s/2} \left| \frac{\Gamma(-\nu)}{\Gamma(-s/2)\Gamma(-\nu+s/2)} \right|, \\ \left| {}_2F_1\left(1+\nu-\frac{s}{2}, 1-\frac{s}{2}; 2+\nu; \frac{a^2}{x^2}\right) \right| &\leq C x^{2(1+\nu)-\text{Re } s} (x^2 - a^2)^{\text{Re } s/2-1-\nu} \\ &\quad \times \left| \frac{\Gamma(2+\nu)}{\Gamma((2-s)/2)\Gamma((2(1+\nu)+s)/2)} \right|. \end{aligned}$$

These estimates allow to prove the convergence of the integral (11) as an improper one. In fact, writing it as

$$\lim_{N \rightarrow \infty} \int_0^N \varphi(\lambda) [J_{\nu}(x\lambda)Y_{\nu+1}(a\lambda) - Y_{\nu}(x\lambda)J_{\nu+1}(a\lambda)] d\lambda,$$

we take φ from the subspace $\mathcal{M}_{1/2,1}^{-1}(L_c) \subset \mathcal{M}_{0,1}^{-1}(L_c)$ with $c = \{s \in \mathbb{C} : -1 < \text{Re } s < 0\}$. So, according to Definition 2 φ is given by integral (8) of some function $\Phi(s)$ from the weighted L_1 -space. Then changing the order of integration by Fubini's theorem for each fixed N and using (14), it becomes

$$\begin{aligned} \int_0^N \varphi(\lambda) [J_\nu(x\lambda)Y_{\nu+1}(a\lambda) - Y_\nu(x\lambda)J_{\nu+1}(a\lambda)] d\lambda &= \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} \Phi(s) F_\nu(x, s) ds \\ &- \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} \Phi(s) \int_N^\infty \lambda^{-s} [J_\nu(x\lambda)Y_{\nu+1}(a\lambda) - Y_\nu(x\lambda)J_{\nu+1}(a\lambda)] d\lambda ds, \quad x > a. \end{aligned}$$

We will prove that

$$\lim_{N \rightarrow \infty} \int_{\mu-i\infty}^{\mu+i\infty} \Phi(s) \int_N^\infty \lambda^{-s} [J_\nu(x\lambda)Y_{\nu+1}(a\lambda) - Y_\nu(x\lambda)J_{\nu+1}(a\lambda)] d\lambda ds = 0, \quad x > a. \quad (17)$$

To do this, we appeal to the asymptotic behavior of Bessel functions at infinity [1], Vol. II to find for fixed $x > a$

$$J_\nu(x\lambda)Y_{\nu+1}(a\lambda) - Y_\nu(x\lambda)J_{\nu+1}(a\lambda) = -\frac{2}{\pi\lambda\sqrt{xa}} \left[\cos(\lambda(x-a)) + O\left(\frac{1}{\lambda}\right) \right], \quad \lambda \rightarrow \infty.$$

Substituting this expression into (17) and integrating by parts in the inner integral with respect to λ it gives

$$\int_N^\infty \lambda^{-s} [J_\nu(x\lambda)Y_{\nu+1}(a\lambda) - Y_\nu(x\lambda)J_{\nu+1}(a\lambda)] d\lambda = O((|s|+1)N^{-\mu-1}),$$

and

$$\begin{aligned} &\left| \int_{\mu-i\infty}^{\mu+i\infty} \Phi(s) \int_N^\infty \lambda^{-s} [J_\nu(x\lambda)Y_{\nu+1}(a\lambda) - Y_\nu(x\lambda)J_{\nu+1}(a\lambda)] d\lambda ds \right| \\ &\leq C N^{-\mu-1} \int_{\mu-i\infty}^{\mu+i\infty} |\Phi(s)| (|s|+1) |ds| \rightarrow 0, \quad N \rightarrow \infty \end{aligned}$$

under assumption $\mu + 1 > 0$. Hence we proved the equality

$$\int_0^\infty \varphi(\lambda) [J_\nu(x\lambda)Y_{\nu+1}(a\lambda) - Y_\nu(x\lambda)J_{\nu+1}(a\lambda)] d\lambda = \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} \Phi(s) F_\nu(x, s) ds, \quad (18)$$

where the integral in the right-hand side of (18) converges absolutely. Indeed, from (14), (16) and Stirling's asymptotic formula for the gamma-function at infinity [1], Vol. I, we have

$$|F(x, s)| \leq C x^{\mu-\text{Re } s} e^{\pi|s|/2} |s|^{-\mu}, \quad x > a.$$

Therefore,

$$\int_{\mu-i\infty}^{\mu+i\infty} |\Phi(s) F_\nu(x, s) ds| \leq C x^{\mu-\text{Re } s} \int_{\mu-i\infty}^{\mu+i\infty} |\Phi(s)| e^{\pi|s|/2} |s| ds = C \|\varphi\|_{\mathcal{M}_{1/2,1}^{-1}(L_c)} x^{\mu-\text{Re } s}.$$

Moreover, it tends to zero when $x \rightarrow \infty$ when $\mu - \text{Re } s < 0$.

Let $f \in \mathcal{M}_{0,1}^{-1}(L_c)$ with $c = \{s \in \mathbb{C} : \text{Re } s = \gamma > 1/2\}$. Returning to integral equation (11) and observing that due to the absolute and uniform convergence of the integral (8) and its derivative with respect to $x \geq x_0 > 0$ functions f, φ are continuously differentiable on $[a, \infty)$ and \mathbb{R}_+ , respectively, we act with the differential operator $[x^\nu \frac{d}{dx} x^{-\nu}]$ on its both sides. Hence, employing (12), (13), we obtain

$$\int_0^\infty \lambda \varphi(\lambda) [Y_{v+1}(x\lambda)J_{v+1}(a\lambda) - J_{v+1}(x\lambda)Y_{v+1}(a\lambda)] d\lambda = \left[x^v \frac{d}{dx} x^{-v} \right] f(x). \quad (19)$$

It is allowed owing to the uniform convergence by $x \geq a_0 > a$ of the integral with respect to λ in (19) if we keep function $\lambda \varphi(\lambda)$ in the same space $\mathcal{M}_{0,1}^{-1}(L_c)$, i.e.

$$\lambda \varphi(\lambda) = \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} \Psi(s) \lambda^{-s} ds, \quad (20)$$

where $|s|\Psi(s) \in L_1(c)$, $c = \{s \in \mathbb{C} : -1 < \mu < 0\}$. In fact, recalling the asymptotic behavior of Bessel functions at infinity and integrating by parts, we find

$$\begin{aligned} & \left| \int_N^\infty \lambda \varphi(\lambda) [Y_{v+1}(x\lambda)J_{v+1}(a\lambda) - J_{v+1}(x\lambda)Y_{v+1}(a\lambda)] d\lambda \right| \\ & \leq \frac{1}{\pi^2 \sqrt{xa}} \left| \int_N^\infty \sin(\lambda(x-a)) \int_{\mu-i\infty}^{\mu+i\infty} \Psi(s) \lambda^{-s-1} ds d\lambda \right| + CN^{-\mu-1} \int_{\mu-i\infty}^{\mu+i\infty} |\Psi(s) ds| \\ & \leq O(N^{-\mu-1}) + \frac{1}{\pi^2(a_0-a)\sqrt{a_0a}} \left| \int_N^\infty \cos(\lambda(x-a)) \int_{\mu-i\infty}^{\mu+i\infty} \Psi(s)(s+1) \lambda^{-s-2} ds d\lambda \right| \\ & \leq O(N^{-\mu-1}) + \frac{N^{-\mu-1}}{\pi^2(a_0-a)\sqrt{a_0a}} \int_{\mu-i\infty}^{\mu+i\infty} |\Psi(s)|(|s|+1) |ds| \rightarrow 0, \quad N \rightarrow \infty, \end{aligned}$$

where the differentiation under the integral sign is allowed via the absolute and uniform convergence.

Meanwhile, by the same reasons for $x \geq a$, we have (see (8))

$$\left[x^v \frac{d}{dx} x^{-v} \right] f(x) = -\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} (s+v) F(s) x^{-s-1} ds, \quad x \geq a,$$

and it tends to zero when $x \rightarrow \infty$ via the estimate

$$\left| \left[x^v \frac{d}{dx} x^{-v} \right] f(x) \right| \leq C x^{-\gamma-1} \|f\|_{\mathcal{M}_{0,1}^{-1}(L_c)}, \quad \gamma > \frac{1}{2}$$

as well as

$$\left| \frac{d}{dx} [x^{-v} f(x)] \right| \leq C x^{-\gamma-v-1} \|f\|_{\mathcal{M}_{0,1}^{-1}(L_c)} \rightarrow 0, \quad x \rightarrow \infty.$$

This means that equations (11), (19) are equivalent. Hence employing Theorem 1, the unique solution of equation (19) has the form

$$\varphi(\lambda) = -\frac{1}{J_{v+1}^2(a\lambda) + Y_{v+1}^2(a\lambda)} \int_a^\infty C_{v+1}(\lambda t, \lambda a) t^{v+1} \frac{d}{dt} [t^{-v} f(t)] dt, \quad \lambda > 0. \quad (20)$$

It can be written in a different form with the integration by parts, eliminating outer integrated terms since $t^{1/2}f(t) = o(1)$, $t \rightarrow \infty$ and taking into account (12), (13). Hence,

$$\begin{aligned} \varphi(\lambda) &= \frac{1}{J_{v+1}^2(a\lambda) + Y_{v+1}^2(a\lambda)} \int_a^\infty t f(t) \left[t^{-1-v} \frac{d}{dt} t^{1+v} \right] C_{v+1}(\lambda t, \lambda a) dt \\ &= \frac{\lambda}{J_{v+1}^2(a\lambda) + Y_{v+1}^2(a\lambda)} \int_a^\infty t f(t) [J_v(\lambda t) Y_{v+1}(a\lambda) - Y_v(\lambda t) J_{v+1}(a\lambda)] dt, \quad \lambda > 0. \end{aligned}$$

We summarize our results by the following

Theorem 2. *Let $-1 < \text{Re } \nu < -1/2$, $f \in \mathcal{M}_{0,1}^{-1}(L_c)$ with $c = \{s \in \mathbb{C} : \text{Re } s = \gamma > 1/2\}$, $\varphi \in \mathcal{M}_{1/2,1}^{-1}(L_c)$ and $\lambda \varphi(\lambda) \in \mathcal{M}_{0,1}^{-1}(L_c)$ with $c = \{s \in \mathbb{C} : -1 < \text{Re } s < \text{Re } \nu\}$. Then φ is the unique solution of the Weber type integral equation (11), given by the formula*

$$\varphi(\lambda) = \frac{\lambda}{J_{\nu+1}^2(a\lambda) + Y_{\nu+1}^2(a\lambda)} \int_a^\infty t f(t) [J_\nu(\lambda t) Y_{\nu+1}(a\lambda) - Y_\nu(\lambda t) J_{\nu+1}(a\lambda)] dt, \lambda > 0.$$

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